LECTURE

Convex Sets and Their Properties

Some Important Definitions

1 **Point Sets** :- Point sets are sets whose elements are points or vectors in E^n (n-dimensional Euclidean space).

(i) A linear equation in two variables x_1, x_2 i.e. $a_1x_1 + a_2x_2 = b$ represents a line in two dimensions. This line may be considered as a set of those points (x_1, x_2) which satisfy $a_1x_1 + a_2x_2 = b$. This sets of points can be written as

$$S_1 = \{(x_1, x_2) : a_1x_1 + a_2x_2 = b\}.$$

(ii) consider the set of points lying inside a circle of unit radius with centre at the origin, in two dimensional space (E^2). Obviously the points (x_1, x_2) of this set satisfy the inequality $x_1^2 + x_2^2 < 1$. This set of points can be written as

$$S_2 = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}.$$

2. **Hypersphere** :- A hypersphere in E^n with centre at a and radius $\epsilon > 0$ is defined to be the sets of points

$$X = \{x : |x - a| = \epsilon\}$$

i.e, the equation of hypersphere in E^n is

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 = \epsilon^2$$

where $a = (a_1, a_2, \ldots, a_n)$, $x = (x_1, x_2, \ldots, x_n)$ which represent a circle in E^2 and sphere in E^3

3. **An-ϵ-neighbourhood** :- An ϵ-neighbourhood about the point a is defined as the set of points lying inside the hypersphere with centre at a and radius $\epsilon > 0$. i.e. the ϵ-neighbourhood about the point a is the set of points

$$X = \{x : |x - a| < \epsilon\}$$

4. **An Interior Point :-** A point a is an interior point of the set S if there exists an ϵ-neighbourhood about a which contains only points of the set S.
An interior points of S must be an element of S.

5. **A Boundary Point :-** A point a is a boundary point of the set S if every ϵ-neighbourhood about a $(\epsilon > 0)$ contains points which are in the set and the points which are not in the set.
A boundary points of S does not have to be an element of S.

6. **An open Set :-** A set S is said to be an open set if it contains only interior points.

7. **A closed Set :-** A set S is said to be a closed set if it contains all its boundary points.

8. **Lines :-** In E^n, the line through the two points x_1 and x_2, $x_1 \neq x_2$ is defined to be the set of points.

$$X = \{x : x = \lambda x_1 + (1-\lambda)x_2 \text{, for all real } \lambda\}$$

9. ~~Line segment :- A hyperplane is defined as the set of points~~

9. **Line Segment :** In E^n, the line segment joining two points x_1 and x_2 is defined to be the set of points -

$$X = \{x : x = \lambda x_1 + (1-\lambda)x_2 \text{, } 0 \leq \lambda \leq 1\}.$$

10. **Hyperplane:-** A hyperplane is defined as the set of points satisfying

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = z \quad (\text{not all } c_i = 0)$$

or

$$cx = z$$

for prescribed values of $c_1, c_2, \ldots c_n$ and z,
The vector c is called vector normal to the hyper-plane and $\pm \frac{c}{|c|}$ are called unit normals.

A hyperplane divides the whole space E^n into three mutually disjoints sets given by

$$X_1 = \{x : cx > z\}$$
$$X_2 = \{x : cx = z\}$$
$$X_3 = \{x : cx < z\}$$

The sets X_1 and X_3 are called open half spaces.
The sets $\{x : cx \leq z\}$ and $\{x : cx \geq z\}$ are called closed half spaces.

11. **Parallel Hyperplanes:-** Two hyperplanes $c_1x = z$ and $c_2x = z$ are said to be parallel if they have the same unit normals i.e, if $c_1 = \lambda c_2$ for some λ, λ being nonzero.

12. **Convex Combination:-** A convex combination of a finite number of points $x_1, x_2, \ldots, x_n$ is defined as a point

$$x = \lambda x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

where λ_i is real and $\geq 0 \ \forall i$

and $$\sum_{i=1}^{n} \lambda_i = 1.$$

The convex combination of two points x_1 and x_2 is given by

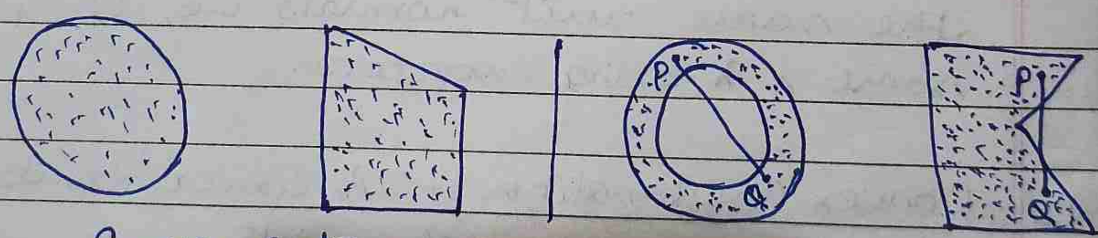
$$x = \lambda_1 x_1 + \lambda_2 x_2 \text{ s.t. } \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1.$$

It can also be written as

$$x = \lambda x_1 + (1-\lambda) x_2, \quad 0 \leq \lambda \leq 1.$$

This show, that the line segment joining two points x_1 and x_2 is nothing but the set of all possible convex combination of the points x_1 and x_2.

13. Convex Set :- A set of points is said to be convex if for any two points in the set, the line joining these two points is also in the set. In other words a set is convex if the convex combination of any two points in the set, is also in the set.



Convex sets Non-Convex sets.

14. Convex Hull :- The convex hull $C(X)$ of any given set of points X is the set of all convex combinations of set of points from X.

example : If X is just the eight vertices of cube, then the convex hull $C(X)$ is the whole cube

15. <u>Convex Function</u> :- A function $f(x)$ is said to be strictly convex at x if for any two other distinct points x_1 and x_2 ;

$$f\{\lambda x_1 + (1-\lambda)x_2\} < \lambda f(x_1) + (1-\lambda) f(x_2),$$

where $0 < \lambda < 1$.

On the other hand a function $f(x)$ is strictly concave if $-f(x)$ is strictly convex.