Important Theorems.

Thm 1 :- A hyperplane is a convex set.

Proof:- Consider the hyperplane

$$X = \{x : cx = z\}$$

Let x_1 and x_2 be any two points in the hyper-plane X

$$\therefore \quad cx_1 = z \quad \text{and} \quad cx_2 = z$$

If $\quad x_3 = \lambda x_1 + (1-\lambda)x_2 \,, \quad 0 \leq \lambda \leq 1$

then $\quad cx_3 = \lambda c x_1 + (1-\lambda) c x_2 = \lambda z + (1-\lambda) z = z$

which implies that

$$x_3 = \lambda x_1 + (1-\lambda) x_2$$ is also point in the hyper-plane X.

Hence by definition, the hyperplane X is a convex set.

Thm 2 :- The closed half spaces $H_1 = \{x : cx \geq z\}$ and $H_2 = \{x : cx \leq z\}$ are convex sets.

Proof :- Let x_1 and x_2 be any two points of H_1.

then $cx_1 \geq z$, $cx_2 \geq z$

Now for

$0 \leq \lambda \leq 1$, $c[\lambda x_1 + (1-\lambda)x_2] = \lambda cx_1 + (1-\lambda)cx_2 \geq \lambda z + (1-\lambda)z$
$= z$

i.e $\forall x_1, x_2 \in H_1$ and $0 \leq \lambda \leq 1 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in H_1$

Hence H_1 is a convex set.

Similarly, replacing the sign $\geq$ by $\leq$ we can prove that H_2 is convex set.

Thm 3 :- Intersection of two convex sets is also a convex set.

Proof :- Consider two convex sets X_1 and X_2. Let X_3 be the intersection of sets X_1 and and X_2 i.e. $X_3 = X_1 \cap X_2$

if $x_1 \in X_1 \cap X_2 \Rightarrow x_1 \in X_1$ and $x_1 \in X_2$

and $x_2 \in X_1 \cap X_2 \Rightarrow x_2 \in X_1$ and $x_2 \in X_2$

Since X_1 and X_2 are convex sets.

$\therefore$ $x_1, x_2 \in X_1 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in X_1$, $0 \leq \lambda \leq 1$

and $x_1, x_2 \in X_2 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in X_2$, $0 \leq \lambda \leq 1$

now $\lambda x_1 + (1-\lambda)x_2 \in X_1$ and $\lambda x_1 + (1-\lambda)x_2 \in X_2$

$\Rightarrow \lambda x_1 + (1-\lambda)x_2 \in X_1 \cap X_2$, $0 \leq \lambda \leq 1$

Hence by def. $X_3 = X_1 \cap X_2$ is a convex set.