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MSc II Sem

(H-2050)

(Measure and Integration)

About Convergence and

Lebsgue Convergence Theorem

→ Convergence in Mean :—

A sequence $\langle f_n \rangle$ of Lebesgue integrable functions is said to converge in mean to a function f if

$$\lim_{n\to\infty} \int_E |f_n - f| \, dx = 0$$

→ Convergence in Measure :—

If $\langle f_n \rangle$ be a sequence of measurable functions defined over a measurable set E. Also f be a measurable function defined over E s.t.

(i) $f(x) < \infty$ a.e. on E

(ii) $\lim_{n\to\infty} m[E \, |f_n - f| \geq \epsilon] = 0 \quad \forall \; \epsilon > 0$

Then $\langle f_n \rangle$ is said to converge in measure to f.

→ Pointwise Convergence :-

Let $\langle f_n \rangle$ be a sequence of measurable functions defined over a measurable set E, If $\exists$ a measurable function f over E s.t.

$$\lim_{n\to\infty} f_n(x) = f(x) \;\; \forall \; x \in E$$

Then $\langle f_n \rangle$ converges pointwise to f on E.

→ Convergent Almost Everywhere :-

Let $\langle f_n \rangle$ be a sequence of measurable functions defined over a measurable set E. If $\exists$ a measurable function f over E and a set A, such tha

(i) $m(A) = 0$

(ii) $\lim_{n\to\infty} f_n(x) = f(x) \;\; \forall \; x \in E - A$

Then $\langle f_n \rangle$ converges to f almost everywhere.

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Lebesgue Convergence theorem :-

Proof :- Given $|f_n(x)| < M \ \forall n \in N, \forall x \in E$ — (1)

Convergence in measure to f

$\Rightarrow \lim_{n\to\infty} m[E(|f_n - f| \geq \epsilon)] = 0,$ — (2)

We show that $\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx$ — (3)

From (1) & (2) we have

$|f(x)| < M$ — (4)

Let $A_n = E(|f_n - f| \geq \delta)$, $B_n = E(|f_n - f| < \delta)$. **

Then $E = A_n \cup B_n$ & $A_n \cap B_n = \phi$.

By convergence in measure, we have

$\lim_{n\to\infty} m(A_n) = 0$ — (5)

By additive property of the integral, we have

$$\int_E |f_n - f|\,dx = \int_{A_n} |f_n - f|\,dx + \int_{B_n} |f_n - f|\,dx \quad \text{— (6)}$$

** we have $|f_n - f| < \delta \ \forall x \in B_n$

$\int_{B_n} |f_n - f|\,dx < \delta \cdot m(B_n) \leq \delta\, m(E)$ [$\because$ by first mean value thm.]

or $\int_{B_n} |f_n - f|\,dx < \delta\, m(E)$ — (7)

Choose $\delta m(E) < \frac{\epsilon}{2}$

$\int_{B_n} |f_n - f|\,dx < \epsilon/2$ — (8)

Now from (1) & (4) we have

$$|f_n - f| \le |f_n| + |f| = M + M = 2M$$

$$\therefore \quad |f_n - f| < 2M$$

Thus, $\displaystyle\int_{A_n} |f_n - f|\,dx < 2M \cdot m(A_n)$ ——— (9) by first mean value theorem

choose $m(A_n) = \dfrac{\epsilon}{4M}$ ——— (10)

$$\int_{A_n} |f_n - f|\,dx < 2M \cdot \frac{\epsilon}{4M} = \frac{\epsilon}{2}$$

$$\Rightarrow \int_{A_n} |f_n - f|\,dx < \frac{\epsilon}{2} \quad \text{——— (11)}$$

Thus, eqn. (6) $\Rightarrow$

$$\int_E |f_n - f|\,dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow \int_E |f_n - f|\,dx < \epsilon$$

Thus $\displaystyle\left|\int_E (f_n - f)\,dx\right| \le \int_E |f_n - f|\,dx < \epsilon$

or $\displaystyle\left|\int_E (f_n - f)\,dx\right| < \epsilon$

$\Rightarrow \displaystyle\lim_{n\to\infty} \int_E (f_n - f)\,dx = 0$ as $\epsilon \to 0$

$$\Rightarrow \lim_{n\to\infty} \int_E f_n(x)\,dx - \int_E f(x)\,dx = 0$$

or $\displaystyle\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx.$

Proved