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MSc II Sem

(H-2050)

(Measure and Integration)

Dominated Convergence Theorem

and

Beppo - Levis Theorem

Theorem 7 169 The Dominated Convergence Theorem :—

Proof :— Given $|f_n(x)| < \psi(x) \ \forall \ x \in E \ \& \ \forall \ n \in N.$ —— (1)

convergence in measure to f

$$\lim_{n\to\infty} m\left[E \, |f_n - f| \geq \epsilon\right] = 0 \quad \text{—— (2)}$$

we show that $\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$

From (1) & (2) we have —— (3)

$$|f(x)| < \psi(x)$$

Given $\psi(x)$ is integrable over E, so $f(x)$ is integrable over E.

Let $A_n = E(|f_n - f| \geq \delta)$, $B_n = E(|f_n - f| < \delta)$ **

Then $E = A_n \cup B_n$, $A_n \cap B_n = \phi$

Convergence in measure

$$\Rightarrow \lim_{n\to\infty} m(A_n) = 0 \quad \text{—— (4)}$$

By additive property of the integral, we have

$$\int_E |f_n - f| \, dx = \int_{A_n} |f_n - f| \, dx + \int_{B_n} |f_n - f| \, dx \quad \text{—— (5)}$$

** we have $|f_n - f| < \delta \ \forall \ x \in B_n$

Thus, $\int_{B_n} |f_n - f| \, dx < \delta \cdot m(B_n) \leq \delta \cdot m(E)$ (using first mean value theorem) —— (6)

Choose δ such that $\delta \cdot m(E) < \frac{\epsilon}{2}$

$$\int_{B_n} |f_n - f| \, dx < \frac{\epsilon}{2} \quad \text{—— (7)}$$

Now $\quad |f_n - f| \le |f_n| + |f| < \psi + \psi = 2\psi \quad$ [From ① & ③]

(27)

Thus, $\quad |f_n - f| < 2\psi \quad \forall\, x \in E$

Integrating, we get

$$\int_{A_n} |f_n - f|\, dx < 2 \int_{A_n} \psi(x)\, dx \quad \text{--- ⑧}$$

Also $\left[|f_n(x)| < \psi(x) \Rightarrow \psi(x) \ge 0 \right.$

$$\Rightarrow \left| \int_{A_n} \psi(x)\, dx \right| = \int_{A_n} \psi(x)\, dx$$

$$\int_{A_n} |f_n - f|\, dx < 2 \left| \int_{A_n} \psi(x)\, dx \right| \quad \text{--- ⑨}$$

Choose $\left| \int_{A_n} \psi(x)\, dx \right| = \frac{\epsilon}{4}$

Then eqn. ⑨ becomes

$$\int_{A_n} |f_n - f|\, dx < 2 \cdot \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

$$\Rightarrow \int_{A_n} |f_n - f|\, dx < \frac{\epsilon}{2} \quad \text{--- ⑩}$$

Eqn ⑤ $\Rightarrow \quad \int_{E} |f_n - f|\, dx < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$$\Rightarrow \int_{E} |f_n - f|\, dx < \epsilon$$

Now $\quad \left| \int_E (f_n - f)\, dx \right| \le \int_E |f_n - f|\, dx < \epsilon$

$$\Rightarrow \left| \int_E (f_n - f)\, dx \right| < \epsilon$$

$$\Rightarrow \lim_{n \to \infty} \int_E (f_n - f)\, dx = 0 \quad \text{as } \epsilon \to 0$$

$$\Rightarrow \lim_{n \to \infty} \int_E f_n(x)\, dx - \int_E f(x)\, dx = 0 \Rightarrow \lim_{n \to \infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx$$

Proved

Theorem 8 / 171 (Beppo-Levi's Theorem) or (... Convergence theorem)

Proof :— we show that

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E \lim_{n\to\infty} f_n(x)\,dx$$

Given $\langle f_n \rangle$ is a non-decreasing sequence i.e.

$$f_1 \le f_2 \le f_3 \le \cdots$$

$$\Rightarrow \quad f_1 \le f_n \ \forall n \ \Rightarrow \ f_n - f_1 \ge 0 \Rightarrow \psi_n \ge 0 \ \forall n$$

where $\psi_n = f_n - f_1$

Given f_n is a sequence of integrable functions, so ψ_n is also a sequence of integrable function.

using bounded convergence theorem, we have

$$\lim_{n\to\infty} \int_E \psi_n\,dx = \int_E \lim_{n\to\infty} \psi_n\,dx$$

$$\Rightarrow \lim_{n\to\infty} \int_E (f_n - f_1)\,dx = \int_E \lim_{n\to\infty} (f_n - f_1)\,dx$$

$$\Rightarrow \lim_{n\to\infty} \int_E f_n\,dx - \cancel{\int_E f_1\,dx} = \int_E \lim_{n\to\infty} f_n\,dx - \cancel{\int_E f_1\,dx}$$

$$\Rightarrow \lim_{n\to\infty} \int_E f_n\,dx = \int_E \lim_{n\to\infty} f_n\,dx .$$