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MSc II Sem

(H-2050)

(Measure and Integration)

Functions of Bounded Variations

Function of Bounded variation —

Consider a real valued function defined on a closed interval $[a, b]$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Define
$$V = \sum_{k=0}^{n-1} | f(x_{k+1}) - f(x_k) |$$

Then V is called the total variation of f and is denoted by $\overset{b}{\underset{a}{V}}(f)$. i.e. $\overset{b}{\underset{a}{V}}(f) = \operatorname{Sup} V$.

If $\overset{b}{\underset{a}{V}} f =$ finite, then f is called a function of finite variation or function of bounded variation.

Thm :— The sum, difference and product of two functions f and g of bounded variations are functions of bounded variations. Hence show that f/g is of bounded variation if $|g(x)| \geq \sigma > 0 \ \forall x$.

Proof :— Let f and g be the functions of bounded variations on $[a, b]$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Then we have
$$\left. \begin{array}{l} \overset{b}{\underset{a}{V}}(f) < \text{finite number} \\ \overset{b}{\underset{a}{V}}(g) < \text{finite number} \end{array} \right\} \text{—— (1)}$$

(i) Let $f + g = s$ we show that s is of bounded variation.

$$|s(x_{k+1}) - s(x_k)| = | [f(x_{k+1}) + g(x_{k+1})] - [f(x_k) + g(x_k)] |$$
$$= | \{ f(x_{k+1}) - f(x_k) \} + \{ g(x_{k+1}) - g(x_k) \} |$$
$$\leq | f(x_{k+1}) - f(x_k) | + | g(x_{k+1}) - g(x_k) |$$

$$\Rightarrow \sum_{k=0}^{n-1} | s(x_{k+1}) - s(x_k) | \leq \sum_{k=0}^{n-1} | f(x_{k+1}) - f(x_k) | + \sum_{k=0}^{n-1} | g(x_{k+1}) - g(x_k) |$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(s) \leq \overset{b}{\underset{a}{V}}(f) + \overset{b}{\underset{a}{V}}(g)$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(s) < \infty \quad (\text{from (1)})$$

$\Rightarrow$ s is of bounded variation.

(ii) Let $d = f - g$

$$|d(x_{k+1}) - d(x_k)| = |[f(x_{k+1}) - g(x_{k+1})] - [f(x_k) - g(x_k)]|$$
$$= |\{f(x_{k+1}) - f(x_k)\} - \{g(x_{k+1}) - g(x_k)\}|$$
$$\leq |f(x_{k+1}) - f(x_k)| + |g(x_{k+1}) - g(x_k)|$$

$$\Rightarrow \sum_{k=0}^{n-1} |d(x_{k+1}) - d(x_k)| \leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| + \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)|$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(d) \leq \overset{b}{\underset{a}{V}}(f) + \overset{b}{\underset{a}{V}}(g)$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(d) < \infty \quad (\text{by } 1)$$

$\Rightarrow$ d is of bounded variation.

(iii) Let $p(x) = f(x)\, g(x)$

Let $A = \sup\{|f(x)| : x \in [a,b]\}$

$B = \sup\{|g(x)| : x \in [a,b]\}$

Now $|p(x_{k+1}) - p(x_k)| = |f(x_{k+1})\, g(x_{k+1}) - f(x_k)\, g(x_k)|$

$$= |f(x_{k+1})\, g(x_{k+1}) - f(x_k)\, g(x_{k+1}) + f(x_k)\, g(x_{k+1}) - f(x_k)\, g(x_k)|$$
$$= |g(x_{k+1})[f(x_{k+1}) - f(x_k)] + f(x_k)[g(x_{k+1}) - g(x_k)]|$$
$$\leq |g(x_{k+1})|\,|f(x_{k+1}) - f(x_k)| + |f(x_k)|\,|g(x_{k+1}) - g(x_k)|$$
$$\leq B\,|f(x_{k+1}) - f(x_k)| + A\,|g(x_{k+1}) - g(x_k)|$$

$$\Rightarrow \sum_{k=0}^{n-1} |p(x_{k+1}) - p(x_k)| \leq B \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| + A \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)|$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(p) \leq B\, \overset{b}{\underset{a}{V}}(f) + A\, \overset{b}{\underset{a}{V}}(g)$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(p) < \infty \quad (\text{from } 1)$$

$\Rightarrow$ p is of bounded variation.

(iv) Let $g(x) \geq \sigma > 0$

Let $h = \frac{1}{g}$

$$\Rightarrow h(x) = \frac{1}{g(x)} \leq \frac{1}{\sigma} > 0$$

$$|h(x_{k+1}) - h(x_k)| = \left| \frac{1}{g(x_{k+1})} - \frac{1}{g(x_k)} \right| = \left| \frac{g(x_k) - g(x_{k+1})}{g(x_k)\, g(x_{k+1})} \right|$$

$$\leq \frac{1}{\sigma^2} \left[g(x_k) - g(x_{k+1}) \right]$$

$$\Rightarrow \sum_{k=0}^{n-1} |h(x_{k+1}) - h(x_k)| \leq \frac{1}{\sigma^2} \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)|$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(h) \leq \frac{1}{\sigma^2} \overset{b}{\underset{a}{V}}(g) < \infty$$

$$\Rightarrow \overset{b}{\underset{a}{V}}(h) < \infty$$

$\Rightarrow$ $h = \frac{1}{g}$ is of bounded variation.

As $f \cdot h = f \cdot \frac{1}{g} = \frac{f}{g}$ is of bounded variation (by result (iii)).

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Now from (iii) we have

$f \cdot h = f \cdot \frac{1}{g} = \frac{f}{g}$ is of bounded variation.

Proved

Theorem :— Every absolutely continuous function is of bounded variation.

Proof :— Let f be an absolutely continuous function defined on $[a,b]$. Let $\delta > 0$ such that $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < 1$ for $\sum_{k=1}^{n} (b_k - a_k) < \delta$.

Divide the interval $[a,b]$ by means of points $a = c_0 < c_1 < c_2 < \cdots < c_{n_0} = b$ in n_0 parts such that $c_{k+1} - c_k < \delta$.

For any subdivision of $[c_k, c_{k+1}]$ we have

$\sum_i |f(x_{i+1}) - f(x_i)| \leq 1$ for $x_{i+1}, x_i \in [c_k, c_{k+1}]$

$\Rightarrow \quad \bigvee_{c_k}^{c_{k+1}} (f) \leq 1$

It follows that

$\bigvee_a^b (f) = \bigvee_{c_0}^{c_1} (f) + \bigvee_{c_1}^{c_2} (f) + \cdots + \bigvee_{c_{n_0-1}}^{c_{n_0}} (f)$

$\leq 1 + 1 + \cdots + 1 = n_0$

i.e. $\bigvee_a^b (f) < \infty \Rightarrow f$ is of bounded variation.