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M.Sc. II Sem
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(Topology)

Neighbourhood and Interior Points.

Thm 26
232

Show that $\overline{A} = A \cup D(A)$

Proof:– First we show that $A \cup D(A)$ is closed.

i.e. $[A \cup D(A)]' = A' \cap D'(A)$ is open.

Let $x \in A' \cap D'(A) \Rightarrow x \in A'$ and $x \in D'(A)$

$\Rightarrow x \notin A$ and $x \notin D(A) \Rightarrow x$ is not a limit point of A $\Rightarrow$ thus $\exists$ an open ~~nbd~~ nbd N_x which contains no point of A i.e. $N_x \subset D'(A)$.

Since $N_x \subset A'$, $N_x \subset D'(A)$

$\Rightarrow N_x \subset A' \cap D'(A)$

$\Rightarrow A' \cap D'(A)$ contains a nbd. of each of its points and so $A' \cap D'(A)$ is open. Consequently $A \cup D(A)$ is closed.

Now we show that $\overline{A} = A \cup D(A)$.

As $A \cup D(A)$ is closed set containing A and $\overline{A}$ is the smallest closed set containing A, we have $\overline{A} \subset A \cup D(A)$. Also $\overline{A}$ is closed, so it contains all its limit points and hence $D(A) \subset \overline{A}$.

Also $A \subset \overline{A}$.

Now we have $A \subset \overline{A}$ & $D(A) \subset \overline{A}$

$\Rightarrow A \cup D(A) \subset \overline{A}$

Hence $\overline{A} = A \cup D(A)$

Thm 27
233

Let $(X, \mathcal{T})$ be a topological space and let A, B be any subsets of X. Then

(i) $\overline{\phi} = \phi$ (ii) $A \subset \overline{A}$

(iii) $A \subset B \Rightarrow \overline{A} \subset \overline{B}$ (iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(v) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ (vi) $\overline{\overline{A}} = \overline{A}$

Proof – (i) we know that ϕ is closed

$\Rightarrow \phi = \overline{\phi}$.

(ii) we know that $\overline{A}$ is the smallest closed set containing A, so $A \subset \overline{A}$.

(iii) We show that $A \subset B \Rightarrow \bar{A} \subset \bar{B}$

Let $A \subset B$

$\Rightarrow A \subset \bar{B}$ (by (ii) as $B \subset \bar{B}$)

$\Rightarrow \bar{A} \subset \bar{B}$ ($\because \bar{A}$ is the smallest closed set containing A i.e. $A \subset \bar{A}$)

Proved

(iv) We show that $\overline{A \cup B} = \bar{A} \cup \bar{B}$

We know that

$A \subset A \cup B$ and $B \subset A \cup B$

$\Rightarrow \bar{A} \subset \overline{A \cup B}$ and $\Rightarrow \bar{B} \subset \overline{A \cup B}$

$\Rightarrow \bar{A} \cup \bar{B} \subset \overline{A \cup B}$ —— (1)

Now, $\bar{A}$ & $\bar{B}$ are closed sets so $\bar{A} \cup \bar{B}$ is also closed.

Again $A \subset \bar{A}$, $B \subset \bar{B}$ $\Rightarrow A \cup B \subset \bar{A} \cup \bar{B}$

$\Rightarrow \bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$.

Since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, so $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ —— (2)

From (1) & (2) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ Proved

(v) We know that

$A \cap B \subset A$ and $A \cap B \subset B$

$\Rightarrow \overline{A \cap B} \subset \bar{A}$ and $\overline{A \cap B} \subset \bar{B}$

$\Rightarrow \overline{A \cap B} \subset \bar{A} \cap \bar{B}$ Proved

(vi) As $\bar{A}$ is the closed set so

$\bar{\bar{A}} = \bar{A}$ (As A is closed if and only if $A = \bar{A}$)

Thm 28 / 234

Let $(X, \mathfrak{J})$ be a topological space and let A be a subset of X. Then the following statements are equivalent

(i) A is closed (ii) $\bar{A} = A$ (iii) A contains all its accumulation points

Proof:- (i) $\Rightarrow$ (ii) Given A is closed

$\Rightarrow A = \bar{A}$

(ii) $\Rightarrow$ (iii) Given $\bar{A} = A \Rightarrow A \cup D(A) = A$ ($\because \bar{A} = A \cup D(A)$)

$\Rightarrow D(A) \subset A \Rightarrow A$ contains all its accmulation points

(iii) $\Rightarrow$ (i) A contains all its accumulation points

$\Rightarrow D(A) \subset A \Rightarrow A \cup D(A) = A \Rightarrow \bar{A} = A$

$\Rightarrow A$ is closed.

Interior Point and Interior of a Set / 241

Let (X, J) be a topological space and let $A \subset X$. Then $x \in A$ is said to be an interior point of A iff A is a nbd. of x i.e. iff $\exists$ an open set G such that $x \in G \subset A$

The set of all interior points of A is called the interior of A and is denoted by A^o or A^i or $i(A)$ or $Int(A)$.

Also A contains all its interior points i.e. $A^o \subset A$.

Thm 30 / 241

Show that
$A^o = \cup\{G : G \text{ is open}, G \subset A\}$

Proof:— Let $x \in A^o \Leftrightarrow A$ is a nbd. of x
$\Leftrightarrow \exists$ an open set G such that $x \in G \subset A$
$\Leftrightarrow x \in \cup\{G : G \text{ is open}, G \subset A\}$
$\Rightarrow A^o = \cup\{G : G \text{ is open}, G \subset A\}$

Thm 31 / 241

Let (X, J) be a topological space and let A be a subset of X. Then
(i) A^o is an open set
(ii) A^o is the largest open set containing in A
(iii) A is open if and only if $A^o = A$

Proof:— (i) Let $x \in A^o \Rightarrow x$ is an interior point of $A \Rightarrow A$ is a nbd. of $x \Rightarrow \exists$ an open set G such that $x \in G \subset A \Rightarrow G$ is open
$\Rightarrow$ every point of G is an interior point of A so $G \subset A^o \Rightarrow$ Thus $x \in A^o$ $\exists$ an open set G

such that $x \in G \subset A^0 \Rightarrow A^0$ is a nbd of each of its points and hence A^0 is open.

(ii) Let $G \subset A$ and $x \in G$. Then $x \in G \subset A$

$\Rightarrow A$ is a nbd. of x and hence x is an interior point of A. Thus $x \in A^0$ so

$x \in G \Rightarrow x \in A^0 \Rightarrow G \subset A^0 \subset A$

$\Rightarrow A^0$ is the largest open set containing A.

(iii) Let $A = A^0$

using (i) A^0 is an open set, therfore A is also an open set $\Rightarrow A$ is open

Conversely, let A is open, we show that $A = A^0$.

Since A is open, then by (ii) A^0 is the largest open subset of A. Hence $A = A^0$.

Thm 33 / 242 Let $(X, \mathcal{J})$ be a topological space and let A, B be any subsets of X. Then

(i) $X^0 = X$, $\phi^0 = \phi$ (ii) $A^0 \subset A$

(iii) $A \subset B \Rightarrow A^0 \subset B^0$ (iv) $(A \cap B)^0 = A^0 \cap B^0$

(v) $A^0 \cup B^0 \subset (A \cap B)^0$ (vi) $A^{00} = A^0$

Proof:— (i) As ϕ & X are open sets, then

$X^0 = X$ & $\phi^0 = \phi$ [As A is open iff $A = A^0$]

(ii) Let $x \in A^0 \Rightarrow A$ is a nbd. of x

$\Rightarrow x \in A$

Thus, $A^0 \subset A$

(iii) Let $x \in A^0 \Rightarrow A$ is a nbd. of x

$\Rightarrow B$ is also a nbd. of x ($\because A \subset B$)

$\Rightarrow B$ is interior point of x

$\Rightarrow x \in B^0$ Thus, $A^0 \subset B^0$

Hence $A \subset B \Rightarrow A^0 \subset B^0$

(iv) we know that

$A \cap B \subset A$ and $A \cap B \subset B$

$(A \cap B)^\circ \subset A^\circ$ $(A \cap B)^\circ \subset B^\circ$

$\Rightarrow$ Thus, $(A \cap B)^\circ \subset A^\circ \cap B^\circ$ —— (1)

Again, we let $x \in A^\circ \cap B^\circ$

$\Rightarrow$ $x \in A^\circ$ and $x \in B^\circ$

$\Rightarrow$ A is a nbd of x and B is a nbd of x

$\Rightarrow$ $A \cap B$ is also a nbd of x

$\Rightarrow$ $A \cap B$ is an interior point of x

$\Rightarrow$ $x \in (A \cap B)^\circ$

$\Rightarrow$ Thus, $A^\circ \cap B^\circ \subset (A \cap B)^\circ$ —— (2)

From (1) & (2) we get

$(A \cap B)^\circ = A^\circ \cap B^\circ$

(v) We know that

$A \subset A \cup B$ and $B \subset A \cup B$

$\Rightarrow$ $A^\circ \subset (A \cup B)^\circ$ & $B^\circ \subset (A \cup B)^\circ$

$\Rightarrow$ $A^\circ \cup B^\circ \subset (A \cup B)^\circ$

(vi) Since A° is open so

$(A^\circ)^\circ = A^\circ$ $\Rightarrow$ $A^{\circ\circ} = A^\circ$

Ref – Krishna Pub Meerut

Exterior of a Set / 245

... Exterior of a ... Let A be