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M.Sc. II Sem

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(Topology)

Exterior Points and Frontier Points

Ref

Exterior Points and Exterior of a Set

Let $(X, \mathfrak{J})$ be a topological space. Let A be a subset of X. Then a point $x \in X$ is said to be an exterior point of A iff $\exists$ an open set G such that $x \in G \subset A'$.

The set of all exterior points of A is called the exterior of A and is denoted by $e(A)$ or A^e or $ext(A)$. Thus, $ext(A) = (A')^\circ$

& $ext(A') = (A'')^\circ = A^\circ$

or in other words

A point $x \in X$ is said to be an exterior point of A iff it is an interior point of the complement A' of A i.e.

$$ext(A) = (A')^\circ$$

Thm 37 / 246 Let $(X, \mathfrak{I})$ be a topological space and let A, B be subsets of X. Then

(i) $ext(X) = \phi$, $ext(\phi) = X$ (ii) $ext(A) \subset A'$

(iii) $ext\, A = ext[(ext(A))']$ (iv) $A \subset B \Rightarrow ext(B) \subseteq ext(A)$

(v) $A^\circ \subset ext(ext(A))$ (vi) $ext(A \cup B) = ext(A) \cap ext(B)$

Proof:- (i)

$ext(X) = (X')^\circ = \phi^\circ = \phi$

& $ext(\phi) = (\phi')^\circ = X^\circ = X$

(ii) $ext(A) = (A')^\circ \subset A'$ $(\because A^\circ \subset A)$

$\Rightarrow ext(A) \subset A'$

(iii) $ext[(ext(A))'] = ext[(A')^{\circ\prime}] = ext(A'^{\circ\prime})$

$= [(A'^{\circ\prime})']^\circ = (A'^{\circ\prime\prime})^\circ = (A'^\circ)^\circ = A'^{\circ\circ}$

$= A'^\circ = (A')^\circ = ext(A)$

[using $A'' = A$, $A^{\circ\circ} = A^\circ$ for any set A]

(iv) $A \subset B \Rightarrow B' \subset A'$

$\Rightarrow (B')^\circ \subset (A')^\circ$

$\Rightarrow ext(B) \subset ext(A)$

(v) From (ii) $ext(A) \subset A'$. Then

(iv) $\Rightarrow$ $ext(A') \subset ext(ext(A))$

$\Rightarrow$ $A^\circ \subset ext(ext(A))$

$(\because ext(A') = [(A')']^\circ = (A'')^\circ = A^\circ)$

(16)

(vi) $ext(A \cup B) = [(A \cup B)']^{\circ}$ (By De Morgan's law)

$= (A' \cap B')^{\circ}$

$= (A')^{\circ} \cap (B')^{\circ}$

$= ext(A) \cap ext(B)$

Frontier Points and Frontier of a Set /248

Let $(X, \mathcal{T})$ be a topological space. A point $x \in X$ is said to be frontier point or boundary point of a subset A of X iff it is neither an interior nor an exterior point of A.

The set of all frontier points of A is called the frontier of A and is denoted by $Fr(A)$.

Thm 40 / 249

Let A be any subset of a topological space X. Then A°, $ext(A)$ and $Fr(A)$ are disjoint and $X = A^{\circ} \cup ext(A) \cup Fr(A)$. Further $Fr(A)$ is closed.

Proof:— We know that

$ext(A) = (A')^{\circ}$

$A^{\circ} \subset A$

$(A')^{\circ} = A'$ $\quad (\because A^{\circ} = A)$

$\phi^{\circ} = \phi$

(*) $Fr(A) = (A^{\circ})' \cap [ext(A)]'$ by De Morgan's law

$= (A^{\circ})' \cap [(A')^{\circ}]'$

$= (A^{\circ})' \cap (A'')^{\circ}$

$Fr(A) = (A^{\circ})' \cap A^{\circ}$

$\Rightarrow Fr(A) \cap A^{\circ} = (A^{\circ})' \cap A^{\circ} \cap A^{\circ}$

$= (A^{\circ})' \cap A^{\circ}$

$= \phi$

as $A' \cap A = \phi$

Since $A \cap A' = \phi \Rightarrow A^{\circ} \cap ext(A) = A^{\circ} \cap (A')^{\circ}$

$= A^{\circ} \cap (A^{\circ})' = \phi$

$\Rightarrow A^{\circ}$ & $ext(A)$ are disjoint

Again, let $x \in Fr(A) \Leftrightarrow x \notin A^{\circ}$ and $x \notin ext(A)$

$\Leftrightarrow x \notin A^{\circ} \cup ext(A)$

$\Leftrightarrow x \in (A^{\circ} \cup ext(A))'$ — (1)

(*) $\Rightarrow Fr(A) = [A^{\circ} \cup ext(A)]'$

$\Rightarrow Fr(A) \cap (A^{\circ}) = \phi$ and $Fr(A) \cap ext(A) = \phi$

and $X = A^{\circ} \cup ext(A) \cup Fr(A)$ and (1) $\Rightarrow A^{\circ}$ & $ext(A)$ are open sets $\Rightarrow Fr(A)$ is closed.