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M.Sc. II Sem

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(Topology)

Continuity in Topological Spaces

Continuity in toplogical space /272

Let (X, J) and (Y, V) be two topological spaces. Let f be a mapping from X to Y. Then f is said to be continuous at $x_0 \in X$ iff $\forall$ V - nbd M of $f(x_0)$ $\exists$ J - nbd N of x_0 such that $f(N) \subset M$

or $\exists$ an open set G_x in X s.t. $x \in G_x \subset f^{-1}[H]$.

Thm 3 / 275 Let X & Y be topological spaces. A mapping $f : X \to Y$ is continuous if and only if the inverse image under f of every open set in Y is open in X.

Proof:- Let X & Y be topological spaces. Let f is continuous, we show that the inverse image of f of every open set in Y is open in X. Let H is open in Y. We show that inverse image of f i.e. $f^{-1}[H]$ is open in X.

If $f^{-1}[H] = \phi$, then f^{-1} is open as ϕ is open. So let $f^{-1}[H] \neq \phi$. Let $x \in f^{-1}[H]$ then $f(x) \in H$. Given f is continuous so $\exists$ an open set G_x in X such that $x \in G_x \subset f^{-1}[H]$

$\Rightarrow$ $f^{-1}[H]$ is a nbd. of each of its points and hence $f^{-1}[H]$ is open in X.

Conversely, let $f^{-1}[H]$ is open in X. Let H is open in Y, We show that f is continuous. Let H is open in Y, and $f(x) \in H$, then $x \in f^{-1}[H]$. Given $f^{-1}[H]$ is open in X. If $f^{-1}[H] = G$ then G is open in X containing H such that $f[G] = f[f^{-1}[H]] \subset H$

$\Rightarrow$ f is continuous function.

Thms 276 Let $(X, \mathcal{T})$ and $(Y, \mathcal{V})$ be two topological spaces. A mapping $f: X \to Y$ is continuous if and only if the inverse image under f of every closed set in Y is closed in X.

Proof:— Let $(X, \mathcal{T})$ & $(Y, \mathcal{V})$ be two topological spaces. Let f is continuous. We show that the inverse image under f of every closed set in Y is closed in X. Let F is closed in Y. Then we show that $f^{-1}[F]$ is closed in X. Given f is continuous and $Y-F$ is open in Y. Then

$$f^{-1}[Y-F] = f^{-1}[Y] - f^{-1}[F] = X - f^{-1}[F] \text{ is open in } X.$$

$\Rightarrow$ $f^{-1}[F]$ is closed in X.

Conversely, let $f^{-1}[F]$ is closed in X. We show that f is continuous. Let F is closed in Y and G is open in Y. Then $Y-G$ is closed in Y and

$f^{-1}[Y-G] = f^{-1}[Y] - f^{-1}[G] = X - f^{-1}[G]$ is closed in X. $\Rightarrow$ $f^{-1}[G]$ is open in X.

Hence f is continuous.

Thm 9 270 A mapping f from a space X into another space Y is continuous if and only if

$$f(\bar{A}) \subset \overline{f(A)} \quad \forall \; A \subset X$$

or

A mapping f is continuous iff for every $x \in X$ arbitrarely closed to A, $f(x)$ is arbitrarily closed to $f(A)$.

or

A map $f: (X, d) \to (Y, \rho)$ be continuous iff

$$f(\bar{A}) \subset \overline{f(A)} \quad \forall \; A \subset X$$

(20)

Proof:— Let f is continuous, we show that $f(\bar{A}) \subset \overline{f(A)}$ $\forall$ $A \subset X$

we know that $\overline{f(A)}$ is closed in X.

$\Rightarrow$ $\overline{f(A)}$ is closed in Y $\Rightarrow f^{-1}[\overline{f(A)}]$ is closed in X

$\Rightarrow$ $\overline{f^{-1}[\overline{f(A)}]} = f^{-1}[\overline{f(A)}]$ —— (1)

Now $f[A] \subset \overline{f[A]}$

$\Rightarrow A \subset f^{-1}\overline{f[A]}$

$\Rightarrow \bar{A} \subset \overline{f^{-1}\overline{f[A]}}$ $\Rightarrow \bar{A} \subset f^{-1}[\overline{f(A)}]$ [from (1)]

$\Rightarrow f[\bar{A}] \subset \overline{f[A]}$ Proved:-

Conversely, let $f(\bar{A}) \subset \overline{f(A)}$ —— (2)

we show that f is continuous.

Let F be closed in Y. Then $F = \bar{F}$

Now, using (2) we have

$f[\overline{f^{-1}(F)}] \subset \overline{f[f^{-1}(F)]} = \bar{F} = F$ (Given)

$\Rightarrow \overline{f^{-1}(F)} \subset f^{-1}(F)$ —— (3)

But $f^{-1}(F) \subset \overline{f^{-1}(F)}$ (always) —— (4)

From (3) & (4) we get

$f^{-1}(F) = \overline{f^{-1}(F)}$

$\Rightarrow f^{-1}(F)$ is closed in X $\Rightarrow f$ is continuous.

Thm 278 A mapping f of a space X onto another space Y is continuous if and only if

$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ $\forall$ $B \subset Y$

Proof:— Let $(X, \mathcal{T})$ and (Y, ρ) be two topological spaces.

Let f is continuous, we show that $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ $\forall$ $B \subset Y$

Since $\overline{B}$ is closed in Y

$\Rightarrow f^{-1}[\overline{B}]$ is closed in X.

$\Rightarrow \overline{f^{-1}[\overline{B}]} = f^{-1}[\overline{B}]$ —— (1)

Now $B \subset \overline{B} \Rightarrow f^{-1}[B] \subset f^{-1}[\overline{B}]$

$\Rightarrow \overline{f^{-1}[B]} \subset \overline{f^{-1}[\overline{B}]}$

$\Rightarrow \overline{f^{-1}[B]} \subset f^{-1}[\overline{B}]$ [from (1)]

Proved

Conversely, let

$\overline{f^{-1}[B]} \subset f^{-1}[\overline{B}]$

We show that f is continuous.

Let F is closed in Y, then $F = \overline{F}$

Also $\overline{f^{-1}[F]} \subset f^{-1}[\overline{F}] = f^{-1}[F]$ —— (2)

But $f^{-1}[F] \subset \overline{f^{-1}[F]}$ (always) — (3)

From (2) & (3) we get

$f^{-1}[F] = \overline{f^{-1}[F]} \Rightarrow f^{-1}[F]$ is closed in X

$\Rightarrow f$ is continuous.

Continuity of Composite Function / 280

Let (X, J), (Y, V) and (Z, W) be three topological spaces and the mapping $f : X \to Y$ and $g : Y \to Z$ be continuous. Then the composite mapping $g \circ f : X \to Z$ is continuous.

Open and Closed Function / 281

Let (X, J) and (Y, V) be two topological spaces. Then a function $f : X \to Y$ is said to be an open function if the image of every open set is open and the function $g : X \to Y$ is said to be a closed function if the image of every closed set is closed.

Ref — Krishna Publication Meerut