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M.Sc IV Sem

(H-4051)

(Functional Analysis)

# Normed Linear Spaces and Banach Spaces

(2)

## Cauchy's Sequence :-

Let $\{x_n\}$ be a sequence in a normed space $(N, \|\,\|)$. Then $\{x_n\}$ is said to be a Cauchy sequence if for each $\epsilon > 0$ $\exists$ a positive integer $M$ such that $\forall\ p, q \geq M$

$$\|x_p - x_q\| < \epsilon$$

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## Banach Space :-

A normed linear space $(X, \|\,\|)$ is called a Banach space if $(X, \|\,\|)$ treated as a metric space is complete. Since every closed subspace of a complete metric space is complete therefore every closed subspace of a Banach space is a Banach space.

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**Question :-** If we have

$$R^n = \{(x_1, x_2, \ldots x_n) : x_1, x_2, \ldots x_n \in X\}.$$

define $\|\,\| : R^n \to R^+ \cup \{0\}$ as follows

$$\|x\| = \left(\sum |x_i - x_{i-1}|^2\right)^{1/2}$$

where $x = (x_1, x_2, \ldots x_n)$

Then $\|\,\|$ is a norm on $R^n$.

$$\|x\| = \left(\sum |x_i|^2\right)^{1/2}$$

**Sol$^n$ :-** (1) obeviously $\|x\| \geq 0$

(2) Let $\|x\| = 0 \Rightarrow \left(\sum |x_i|^2\right)^{1/2} = 0$

$\Rightarrow \sum |x_i|^2 = 0 \Rightarrow |x_i|^2 = 0 \quad \forall\ i = 1, 2, \ldots n$

$\Rightarrow |x_i| = 0 \Rightarrow x = 0$

Conversely, let $x = 0 \Rightarrow x = (0, 0, \ldots 0)$

$\Rightarrow \sum |x_i|^2 = (0, 0, \ldots 0) \Rightarrow \left(\sum |x_i|^2\right)^{1/2} = (0, 0, \ldots 0)$

$\Rightarrow \|x\| = 0$

(3) Let $\alpha \in R$, $x \in R^n$

$x = (x_1, x_2, \ldots x_n)$

$\alpha x = (\alpha x_1, \alpha x_2, \ldots \alpha x_n)$

$$\|\alpha x\| = \left(\sum_{i=1}^{n} |\alpha x_i|^2\right)^{1/2}$$

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$$= \left( \sum_{i=1}^{n} |\alpha|^2 |x_i|^2 \right)^{1/2} = |\alpha| \left( \sum |x_i|^2 \right)^{1/2}$$
$$= |\alpha| \, \|x\|$$

(4) Let $x, y \in R^n$ then

$x = (x_1, x_2, \cdots x_n)$, $y = (y_1, y_2, \cdots y_n)$

$x + y = (x_1 + y_1, x_2 + y_2, \cdots x_n + y_n)$

$$\|x+y\| = \left( \sum_{i=1}^{n} |x_i + y_i|^2 \right)^{1/2}$$
$$\leq \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2} + \left( \sum_{i=1}^{n} |y_i|^2 \right)^{1/2} \quad \text{(By Schwarz's inequality)}$$
$$\leq \|x\| + \|y\|$$

Hence $(R^n, \| \ \|)$ is a normed linear space.

We further claim that <u>$R^n$ is a Banach space.</u>

Let $\{x^k\}$ be a Cauchy sequence in $R^n$. Therefore given $\epsilon > 0$ $\exists$ a positive integer $N$ such that $\forall$ $k, l \geq N$, $\|x^k - x^l\| < \epsilon$.

Since $x^k \in R^n$ $\therefore$ $x^k = (x_1^k, x_2^k, \cdots x_n^k)$
$x^l = (x_1^l, x_2^l, \cdots x_n^l)$

$$\therefore \|x^k - x^l\| = \left( \sum_{i=1}^{n} |x_i^k - x_i^l|^2 \right)^{1/2}$$

$$\Rightarrow \left( \sum_{i=1}^{n} |x_i^k - x_i^l|^2 \right)^{1/2} < \epsilon$$

$\Rightarrow$ $|x_i^k - x_i^l| < \epsilon$ $\Rightarrow$ $\{x_i^k\}$ is a Cauchy sequence in $R$.

$\therefore$ $\{x_i^k\}$ converges in $R$.

Let $x_i^k \longrightarrow x_i$

consider $x = (x_1, x_2, \cdots x_n)$

we claim that $\{x^k\} \rightarrow x$

Since $x_i^k \rightarrow x_i$, given $\epsilon > 0$ $\exists$ a positive integer $k_0(i)$ such that

$$|x_i^k - x_i| < \frac{\epsilon}{\sqrt{n}} \quad \forall\ k > k_0(i)$$

Let $k_0 = \max.\ \{k_0(1), k_0(2), \cdots k_0(n)\}$. Then

$$|x_i^k - x_i| < \epsilon/\sqrt{n} \quad \forall\ k \geq k_0$$

$$|x_i^k - x_i|^2 < \frac{\epsilon^2}{n} \Rightarrow \sum_{i=1}^{n} |x_i^k - x_i|^2 < n \cdot \frac{\epsilon^2}{n} = \epsilon^2$$

$$\Rightarrow \left(\sum_{i=1}^{n} |x_i^k - x_i|^2\right)^{1/2} < \epsilon \quad \Rightarrow \quad \|x^k - x\| < \epsilon$$

Let $\epsilon \to 0$ then $x^k \to x$ and hence $R^n$ becomes a Banach space.

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Question:— Let $S$ be any set and let $f_b(S)$ be the vector space of all bounded scalar valued function on $S$. Let $f, g \in f_b(S)$ and

$$(f+g)(x) = f(x) + g(x)$$

Let $\alpha \in F$, $f \in f_b(S) \Rightarrow (\alpha f)(x) = \alpha f(x)$

Now for $f \in f_b(S)$ define

$$\|f\| = \sup_{s \in S} |f(s)|$$ Show that $\|f\|$ is a norm.

Soln:— We claim that $\|\ \|$ is a norm on $f_b(S)$.

(1) By the defining property of $\|\ \|$

$$\|f\| > 0 \ \forall\ f \in f_b(S)$$

(2) Let $\|f\| = 0 \Rightarrow \sup_{s \in S} |f(s)| = 0 \Rightarrow |f(s)| = 0$

$$\Rightarrow f(s) = 0 \ \forall\ s \in S \Rightarrow f = 0$$

Conversely, let $f \in f_b(S)$ be such that $f = 0$

Then $f(s) = 0 \ \forall\ s \in S$

$$\Rightarrow |f(s)| = 0 \Rightarrow \sup_{s \in S} |f(s)| = 0 \Rightarrow \|f\| = 0$$

Thus, $\|f\| = 0 \Leftrightarrow f = 0$

(3) Let $\lambda \in F$, $f \in f_b(S)$ Then we show that

$$\|\lambda f\| = |\lambda|\, \|f\|$$

Now $$\|\lambda f\| = \sup_{s \in S} |(\lambda f)(s)| = \sup_{s \in S} |\lambda f(s)|$$

$$= \sup_{s \in S} |\lambda|\, |f(s)| = |\lambda| \sup_{s \in S} |f(s)|$$

$$= |\lambda|\, \|f\|$$

(4) Let $f, g \in Fb(S)$ we show that
$$\|f+g\| \leq \|f\| + \|g\|$$
we have
$$\|f+g\| = \sup_{s \in S} |(f+g)(s)| = \sup_{s \in S} |f(s)+g(s)|$$
$$\leq \sup_{s \in S} [\,|f(s)| + |g(s)|\,]$$
$$= \sup_{s \in S} |f(s)| + \sup_{s \in S} |g(s)| = \|f\| + \|g\|$$
So all the conditions are satisfied, and hence $(Fb(S), \|\ \|)$ is a normed linear space.
Now, we show that $(Fb(S), \|\ \|)$ is a Banach space. Let $\{f_n\}$ be a Cauchy sequence in $\{Fb(S), \|\ \|\}$. Therefore given $\epsilon > 0$ $\exists$ a positive integer $N$ such that
$$\|f_n - f_m\| < \epsilon \quad \forall\ n, m \geq N$$
Now $\|f_n - f_m\| = \sup_{s \in S} |(f_n - f_m)(s)|$
$$= \sup_{s \in S} |f_n(s) - f_m(s)|$$
Thus, $\|f_n - f_m\| < \epsilon \Rightarrow \sup_{s \in S} |f_n(s) - f_m(s)| < \epsilon$
$$\Rightarrow |f_n(s) - f_m(s)| < \epsilon \quad \forall\ n, m \geq N,\ s \in S$$
Therefore $f_n(s)$ is a Cauchy sequence in $R$ (or $C$).
Since $R$ is complete, therefore $\{f_n(s)\}$ converges.
Let $\{f_n(s)\} \to f \in R$. Since $f$ depends upon $s$, so let $f = f(s)$

$\therefore$ for each $s \in S$, $\lim_{n \to \infty} f_n(s) = f(s)$

We claim that $\{f_n\} \to f$.
First we show that $f$ is bounded, therefore there exists $k$ such that $|f_n| < k \ \forall n$
Now $|f(x)| = |\lim_{n \to \infty} f_n(x)| = \lim_{n \to \infty} |f_n(x)| < k \quad \forall x$
therefore $f$ is bounded.
Let $m \to \infty$, keeping $n$ fixed in $\|f_n - f_m\|$, we get $\|f_n - f_m\| = \|f_n - f\| \ \forall\ n \geq N$
$$\Rightarrow \|f_n - f\| < \epsilon \ \forall\ n \geq N \Rightarrow \{f_n\} \to f$$
So every Cauchy seq. converges.
Hence $\{Fb(S), \|\ \|\}$ is a Banach space

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**Question :-** Let $X = \ell^p$ for $1 \le p \le \infty$ where $\ell^p$ is the space of all sequences in R (or C), and $\{x_1, x_2, \cdots x_n\}$ such that

$$\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} < \infty$$

For any $x \in X$, define

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \quad \text{where } x = \{x_i\}_{i \in N}$$

Then prove that $(\ell^p, \| \|_p)$ is a norm on X.

**Proof :-** (i) By the defining the property of $\| \|_p$,

$$\|x\|_p > 0 \ \forall \ x \in \ell^p$$

(ii) Let $\|x\|_p = 0$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} = 0 \Rightarrow \sum_{i=1}^{\infty} |x_i|^p = 0 \quad \forall \ i = 1, 2, \cdots \infty$$

$$\Rightarrow |x_i|^p = 0 \Rightarrow |x_i| = 0 \Rightarrow x_i = 0 \Rightarrow x = 0$$

conversely, let $x = 0$

$\Rightarrow \quad x = (0, 0, \cdots 0)$ since $(\sum |0|^p)^{1/p} = 0$

$\Rightarrow \quad \|x\|_p = 0$

(iii) Let $\alpha \in R$ and $x \in \ell^p$

$x \in \ell^p \Rightarrow x = (x_1, x_2, \cdots x_n)$

$\therefore \quad \alpha x = (\alpha x_1, \alpha x_2, \cdots \alpha x_n)$

So $$\|\alpha x\|_p = \left(\sum_{i=1}^{\infty} |\alpha x_i|^p\right)^{1/p} = \left(\sum_{i=1}^{\infty} |\alpha|^p |x_i|^p\right)^{\frac{1}{p}}$$

$$= |\alpha| \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} = |\alpha| \, \|x\|_p$$

(iv) Let $x, y \in \ell^p$, then

$x = (x_1, x_2, \cdots x_n)$ , $y = (y_1, y_2, \cdots y_n)$

we show that $\|x + y\| \le \|x\| + \|y\|$

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots x_n + y_n)$$

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$$\|x+y\| = \left(\sum_{i=1}^{\infty} |x_i+y_i|^p\right)^{1/p}$$

$$\leq \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{1/p}$$

(by Schwarz's inequality)

$$\leq \|x\| + \|y\|$$

Hence $(\ell^p, \|\cdot\|_p)$ is a normed linear space.

Now we show that $(\ell^p, \|\ \|_p)$ is a Banach space.

Let $\{x^k\}$ is a Cauchy sequence in $\ell^p$, then for $\epsilon > 0$ $\exists$ a positive integer $N$ such that

$\|x^k - x^\ell\|_p < \epsilon \quad \forall\ k, \ell \geq N$

Let $x^k = \{x_1^k, x_2^k, \ldots x_n^k, \ldots\}$

$x^\ell = \{x_1^\ell, x_2^\ell \ldots\}$

$\therefore \|x^k - x^\ell\|_p < \epsilon$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |x_i^k - x_i^\ell|^p\right)^{1/p} < \epsilon \Rightarrow \sum_{i=1}^{\infty} |x_i^k - x_i^\ell|^p < \epsilon^p$$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i^k - x_i^\ell|^p < \epsilon^p \quad \text{——— (1)}$$

$$\Rightarrow \|x_i^k - x_i^\ell\| < \epsilon \quad \forall i$$

$\Rightarrow x_i^k$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete.

let $x_i^k \to x_i \in \mathbb{R}\ \forall i$

Let $x = \{x_1, x_2, \ldots\}$. Then $x \in \ell^p$ and we claim that $\{x^k\} \to x$.

From eqn (1) $\sum_{i=1}^{\infty} |x_i^k - x_i^\ell|^p < \epsilon^p$

let $\ell \to \infty$ then $\sum_{i=1}^{\infty} |x_i^k - x_i|^p < \epsilon^p \quad \forall i$

$$\Rightarrow \sum_{i=1}^{\infty} |x_i^k - x_i|^p < \epsilon^p \Rightarrow \left(\sum_{i=1}^{\infty} |x_i^k - x_i|^p\right)^{1/p} < \epsilon$$

$$\Rightarrow \|x^k - x\| < \epsilon \Rightarrow x^k \to x$$

$\Rightarrow (\ell^p, \|\ \|_p)$ is a Banach space.

Ref — Krishna Pub. Text book Meerut.