Dr. S.K. Rana

Deptt. of Mathematics

M.Sc IV Sem

(H-4051)

(Functional Analysis)

Open Mapping Theorem
and
Inverse Mapping Theorem

Open mapping Lemma :—

Let X & Y be Banach spaces over the field K and let $T : X \xrightarrow{\text{onto}} Y$ be a bounded linear transformation. Let
$B_n = B(0, 2^{-n})$ and $B_n' = B(0, \frac{\epsilon}{2^n})$ be open balls in X & Y. Then for some $\epsilon > 0$
$B_1' \subset T(B_0)$

Open Mapping Theorem :—

Let X & Y be Banach spaces and let $T \in B(X, Y)$. Then T is open i.e. every continuous linear transformation is open.

Proof :— Let U be any open subset in X, we show that $T(U)$ is open in Y. Let $y \in T(U)$. Since T is onto $\exists\ x \in U$ such that $y = T(x)$. Now $x \in U$ and U is open therefore $\exists$ an open ball say $B(x, r)$ such that $x \in B(x, r) \subset U$

$\Rightarrow \quad x + rB(0,1) \subset U$

$\Rightarrow \quad x + rB_0 \subset U \quad \Rightarrow \quad B_0 \subset \frac{1}{r}(U - x)$ ——— (1)

From (1) we have

$T(B_0) \subset T\left(\frac{1}{r}(U - x)\right)$

$= \frac{1}{r}[T(U) - T(x)]$ $\quad$ ($\because T$ is linear)

$= \frac{1}{r}[T(U) - y]$ ——— (2)

Now, from open mapping lemma, we have

$B_1' \subset T(B_0)$

from (2) we have $\quad B_1' \subset \frac{1}{r}(T(U) - y)$

$\Rightarrow \quad rB_1' + y \subset T(U)$

$\Rightarrow \quad B(y, \frac{r\epsilon}{2}) \subset T(U)$

$\left[\begin{array}{l} B_1' = B(0, \frac{\epsilon}{2}) \\ rB_1' = B(0, \frac{r\epsilon}{2}) \\ y + rB_1' = B(y, \frac{r\epsilon}{2}) \end{array}\right.$

$\therefore T(U)$ is a nbd. of y, since y is an arbitrary element of $T(U)$. Therefore $T(U)$ is open in Y and hence T is open.

Inverse Mapping Theorem :—

If T is a one to one bounded linear function from a Banach space X onto a Banach space Y, then T^{-1} exists and is bounded.

Proof :— Given $T: X \to Y$ is such that it is one one and onto, therefore $T^{-1}: Y \to X$ exists. Also T^{-1} is one one and onto. We claim that T^{-1} is linear.

Let $y_1, y_2 \in Y$ & $\alpha, \beta \in K$. Therefore there exists $x_1, x_2 \in X$ such that

$T(x_1) = y_1$, & $T(x_2) = y_2$

$$\therefore \alpha y_1 + \beta y_2 = \alpha T(x_1) + \beta T(x_2)$$
$$= T(\alpha x_1 + \beta x_2) \quad (\because T \text{ is linear})$$

$$\Rightarrow T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2$$
$$= \alpha T^{-1}(y_1) + \beta T^{-1}(y_2)$$

$\Rightarrow T^{-1}$ is linear.

Further since T is bounded, by the open mapping lemma T is open. We claim that T^{-1} is bounded. For this, we show that T^{-1} is continuous. Let U be any open subset of X. Then $T(U)$ is open subset of Y. But we have

$$T(U) = (T^{-1})^{-1}(U)$$

$\Rightarrow$ inverse image of open set under T^{-1} is open subset of Y.

$\Rightarrow T^{-1}$ is continuous and hence bounded.

Ref- Krishna Pub Meerut.