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M.Sc IV Sem
(H-4051)
(Functional Analysis)

Inner Product Spaces

$\Rightarrow f$ is bounded.
$\Rightarrow f^{-1}$ is bounded (By Inverse mapping theorem)

$$\begin{aligned}\|T(x)\| &\le \|x\| + \|T(x)\| \\ &= \|(x, T(x))\| \\ &= \|f^{-1}(x)\| \\ &\le \|x\| \quad (\because f^{-1} \text{ is bounded}) \\ &\qquad\qquad \text{So } \|f^{-1}(x)\| \le \|x\|\end{aligned}$$

$\Rightarrow T$ is bounded.

Inner Product Space :—

Let V be a vector space over the field of complex number K, define a function $\langle\ \rangle : V \times V \to K$ such that for each $\alpha, \beta \in K$, $u, v, w \in V$

(i) $\langle \alpha u + \beta v, w \rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

(ii) $\langle u, v \rangle = \overline{\langle v, u\rangle}$

(iii) $\langle u, u \rangle \geq 0$

(iv) $\langle u, u \rangle = 0$ iff $u = 0$

Then $\langle , \rangle$ is called an inner product on V. And the space $(V, \langle , \rangle)$ is called an inner product space.

Remarks :—

(i) $\langle 0, x \rangle = 0 \ \forall \ x \in N$

This follows from
$\langle 0, x \rangle = \langle 0+0, x \rangle = \langle 0, x\rangle + \langle 0, x\rangle$
$\Rightarrow \langle 0, x \rangle = 0$

(ii) If $\langle y, x \rangle = 0 \ \forall \ y \in V$ then $x = 0$

Since $\langle y, x \rangle = 0 \ \forall \ y \in V$ (in particular) we have $\langle x, x \rangle = 0$ which in view of (iv) implies that $x = 0$

(iii) If $\langle y, x \rangle = \langle z, x \rangle$ then $y = z$

$\langle y, x \rangle = \langle z, x \rangle \ \forall \ x \in V$
$\Rightarrow \langle y, x \rangle - \langle z, x \rangle = 0 \ \forall \ x \in V$

$\Rightarrow \quad \langle y-z, x\rangle = 0 \quad \forall\, x \in V$

$\Rightarrow \quad y-z=0 \quad \Rightarrow \quad y=z$

(iv) $\langle u, \alpha v+\beta w\rangle = \bar{\alpha}\langle u,v\rangle + \bar{\beta}\langle u,w\rangle$

$$\begin{aligned} \text{LHS} &= \langle u, \alpha v+\beta w\rangle \\ &= \overline{\langle \alpha v+\beta w, u\rangle} \\ &= \overline{\alpha\langle v,u\rangle + \beta\langle w,u\rangle} \\ &= \overline{\alpha\langle v,u\rangle} + \overline{\beta\langle w,u\rangle} \\ &= \bar{\alpha}\,\overline{\langle v,u\rangle} + \bar{\beta}\,\overline{\langle w,u\rangle} \\ &= \bar{\alpha}\langle u,v\rangle + \bar{\beta}\langle u,w\rangle \end{aligned}$$

Cauchy Schwarz Inequality :—

Let $(V, \langle\ \rangle)$ be an inner product space, then for any $u, v \in V$

$$|\langle u,v\rangle|^2 \leq \langle u,u\rangle\langle v,v\rangle$$

Proof :— If either $u=0$ or $v=0$ then the inequality is trivially true. Hence let us assume that both $u \neq 0$, $v \neq 0$

Now for any $\alpha \in K$, we have

$$\begin{aligned} 0 &\leq \langle u-\alpha v, u-\alpha v\rangle \\ &= \langle u, u-\alpha v\rangle - \alpha\langle v, u-\alpha v\rangle \\ &= \langle u,u\rangle - \bar{\alpha}\langle u,v\rangle - \alpha\langle v,u\rangle + \alpha\bar{\alpha}\langle v,v\rangle \\ &= \langle u,u\rangle - \bar{\alpha}\langle u,v\rangle - \alpha\left[\langle v,u\rangle - \bar{\alpha}\langle v,v\rangle\right] \end{aligned}$$

Choose $\alpha \in K$ such that

$$\bar{\alpha} = \frac{\langle v,u\rangle}{\langle v,v\rangle}$$

Hence, we get

$$0 \leq \langle u,u\rangle - \frac{\langle v,u\rangle}{\langle v,v\rangle}\langle u,v\rangle - \alpha\left[\langle v,u\rangle - \frac{\langle v,u\rangle}{\cancel{\langle v,v\rangle}}\cancel{\langle v,v\rangle}\right]$$

$$\Rightarrow 0 \leq \langle u,u\rangle - \frac{\overline{\langle u,v\rangle}\langle u,v\rangle}{\langle v,v\rangle}$$

or $$\frac{\overline{\langle u,v\rangle}\langle u,v\rangle}{\langle v,v\rangle} \leq \langle u,u\rangle$$

$$\Rightarrow |\langle u,v\rangle|^2 \leq \langle u,u\rangle\langle v,v\rangle \quad (\because |z|^2 = z\bar{z})$$

Hence Proved

Corollary :— If $(V, \langle \, \rangle)$ is an inner product space and if $u, v \in V$ then

$$(\langle u+v, u+v\rangle)^{1/2} \leq \langle u,u\rangle^{1/2} + \langle v,v\rangle^{1/2}$$

Proof :— We have

$$\langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$
$$= \langle u,u\rangle + \langle u,v\rangle + \langle \overline{u,v}\rangle + \langle v,v\rangle$$
$$= \langle u,u\rangle + 2\,\mathrm{Re}\langle u,v\rangle + \langle v,v\rangle$$
$$\leq \langle u,u\rangle + 2|\langle u,v\rangle| + \langle v,v\rangle$$
$$\leq \langle u,u\rangle + 2\langle u,u\rangle^{1/2}\langle v,v\rangle^{1/2} + \langle v,v\rangle$$

(by Cauchy's Schwarz inequality)

$$= (\langle u,u\rangle^{1/2} + \langle v,v\rangle^{1/2})^2$$

$$\Rightarrow \langle u+v, u+v\rangle^{1/2} = \langle u,u\rangle^{1/2} + \langle v,v\rangle^{1/2}$$

Proved

If $z = x+iy$ then $\bar{z} = x-iy$

$z+\bar{z} = 2x$
$= 2\,\mathrm{Re}\,z$

Definitions :— Let $(V, \langle \, \rangle)$ be an inner product space. Define a function $\| \; \| : V \to K$ such that $\|v\| = \sqrt{\langle v,v\rangle}$ $\forall\, v \in V$.

Then $\| \; \|$ is a norm on V.

Proof :— (i) Let $v \neq 0$ then $\|v\| = \sqrt{\langle v,v\rangle} > 0$ Since $\langle v,v\rangle > 0$

(ii) Let $\|v\| = 0 \Leftrightarrow \sqrt{\langle v,v\rangle} = 0$

$\Leftrightarrow \langle v,v\rangle = 0 \Leftrightarrow v = 0$

(iii) Let $v \in V$ & $\alpha \in K$ Then

$$\|\alpha v\| = \sqrt{\langle \alpha v, \alpha v\rangle} = \sqrt{\alpha\bar{\alpha}\langle v,v\rangle}$$
$$= \sqrt{|\alpha|^2\langle v,v\rangle} = |\alpha|\sqrt{\langle v,v\rangle} = |\alpha|\,\|v\|$$

(iv) Let $u, v \in V$. Then

$$\|u+v\| = \sqrt{\langle u+v, u+v\rangle} \leq \sqrt{\langle u,u\rangle} + \sqrt{\langle v,v\rangle}$$
$$= \|u\| + \|v\|$$

So Cauchy Schwarz inequality can be written as

$$|\langle v,w\rangle| \leq \|v\|\,\|w\|$$

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Theorem :- In an inner product space $\langle V, \langle , \rangle \rangle$ the parallelogram law holds i.e. for $u, v \in V$, we have

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

Proof :-

$$LHS = \|u+v\|^2 + \|u-v\|^2$$
$$= \langle u+v, u+v \rangle + \langle u-v, u-v \rangle$$
$$= \left[\langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle\right]$$
$$+ \left[\langle u,u \rangle - \langle u,v \rangle - \langle v,u \rangle + \langle v,v \rangle\right]$$
$$= 2\left[\langle u,u \rangle + \langle v,v \rangle\right] = 2\left[\|u\|^2 + \|v\|^2\right]$$
$$= R.H.S.$$

Theorem :- The inner product is jointly continuous in an inner product space i.e. if $u_n \to u$, $v_n \to v$ then $\langle u_n, v_n \rangle \to \langle u, v \rangle$

Proof :- Let $\epsilon > 0$. Since $\langle u_n \rangle \to u$, we can find a +ive integer N such that

$$\|u_n - u\| < \epsilon/2 \quad \forall\ n \geq N$$

Similarly, since $\langle v_n \rangle \to v$ we can find a +ive integer M such that

$$\|v_n - v\| < \epsilon/2 \quad \forall\ n \geq M$$

Let $R = \min\{M, N\}$

Consider now $|\langle u_n, v_n \rangle - \langle u, v \rangle|$

$$= |\langle u_n, v_n \rangle - \langle u_n, v \rangle + \langle u_n, v \rangle - \langle u, v \rangle|$$
$$= |\langle u_n, v_n - v \rangle + \langle u_n - u, v \rangle|$$
$$\leq |\langle u_n, v_n - v \rangle| + |\langle u_n - u, v \rangle|$$
$$\leq \|u_n\|\,\|v_n - v\| + \|u_n - u\|\,\|v\|$$ (By Cauchy Schwarz inequality)
$$\leq \|u_n\|\,\epsilon/2 + \epsilon/2\,\|v\|$$
$$\to 0 \text{ as } \epsilon \to 0$$
$$\Rightarrow \langle u_n, v_n \rangle \to \langle u, v \rangle$$

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