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M.Sc IV Sem

(H-4051)

(Functional Analysis)

Hahn Banach Theorem

Hahn Banach Lemma :— (Statement)

Let X be a real linear space and let p be a sublinear functional on X. If f is a real linear functional defined on a subspace M of x satisfying $f(x) \leq p(x)$ $\forall\, x \in M$. Then $\exists$ a real linear functional f_0 on X such that $f_0|_M = f$ and $f_0(x) \leq p(x)$ $\forall\, x \in X$.

Hahn Banach Theorem :—

Let X be a normed space over the field C and let M be a subspace of X. Then for every bounded linear functional g on X such that $g|_M = f$ and $\|g\|_X = \|f\|_M$

Proof :— **Case I :—** Let X be a normed space over the field of reals.

Define a function $p : X \to R^+ \cup \{0\}$ such that

$$p(x) = \|f\|_M \|x\| \quad \forall\, x \in X$$

Step 1 :— We claim that p is sublinear.

Let $x, y \in X$ and $\alpha \in R$. Then

$$p(x+y) = \|f\|_M \|x+y\| \leq \|f\|_M (\|x\| + \|y\|)$$

$$= \|f\|_M \|x\| + \|f\|_M \|y\| \leq p(x) + p(y)$$

~~and $p(\alpha x) = \|f\|_M \|x+y\| \leq \|f\|_M (\|x\| + \|y\|)$~~

~~$= \|f\|_M \|x\| + \|f\|_M \|y\|$~~

~~$\leq p(x) + p(y)$~~

and $p(\alpha x) = \|f\|_M \|\alpha x\| = \|f\|_M |\alpha| \|x\|$

$$= \alpha \|f\|_M \|x\| = \alpha p(x)$$

Thus, p is sublinear function on X.

Step 2 :— We show that $g|_M = f$

Also we have

$f(x) \leq |f(x)| \leq \|f\| \|x\|$ ($\because f$ is bounded)

(28)

and $-f(x) \le |f(x)| \le \|f\| \, \|x\|$

$\therefore \; |f(x)| \le \|f\| \, \|x\| = p(x) \; \forall \; x \in M.$

Therefore by the Hahn Banach Lemma there exists a linear functional g on X such that $g|M = f$ and $g(x) \le p(x) \; \forall \; x \in X.$

g is our required function

Step 3 :- We show that g is our required function and g is bounded.

Since $g(x) \le p(x) = \|f\|_M \|x\| \; \forall \; x \in X.$ —— (1)

and $-g(x) = g(-x) \le p(-x)$

$= \|f\|_M \|-x\| = \|f\|_M \|x\|$ —— (2)

Hence from (1) & (2) we have

$|g(x)| \le \|f\|_M \|x\| \; \forall \; x \in X \Rightarrow g$ is bounded.

Step 4 :- We show that $\|g\|_X = \|f\|_M.$

From $|g(x)| \le \|f\|_M \|x\|$, we get

$$\frac{|g(x)|}{\|x\|} \le \|f\|_M \; \forall \; x \in X$$

$$\Rightarrow \sup_{x \in X} \left(\frac{|g(x)|}{\|x\|} \right) \le \|f\|_M$$

$$\Rightarrow \|g\|_X \le \|f\|_M \quad \text{—— (3)}$$

In order to show that the reverse inequality, Let $x = x_1 \in M$ be such that $x_1 \neq 0$. Then

$$\|g\|_X \ge \frac{|g(x)|}{\|x\|} = \frac{|f(x_1)|}{\|x_1\|} \quad [\because g|M = f]$$

$$\Rightarrow \|g\|_X \ge \sup_{x_1 \in M} \left(\frac{|f(x_1)|}{\|x_1\|} \right) = \|f\|_M$$

$$\Rightarrow \|g\|_X \ge \|f\|_M \quad \text{—— (4)}$$

Hence from (3) & (4) we get

$$\|g\|_X = \|f\|_M.$$

P.T.O. $\Rightarrow$

Case 2. Let N be a normed space over the field of complex numbers. When N is a normed space over the field of complex numbers then N is a subspace over the field of complex numbers and $f \in M^*$ is a complex valued functional on M. For any $x \in M$, let

$$f(x) = f_1(x) + i f_2(x)$$

where f_1 & f_2 are real valued functions.

We claim that f_1 & f_2 are linear.

Step 1 - Let $x, y \in M$. Then

$$f(x+y) = f_1(x+y) + i f_2(x+y) \quad (\because f \text{ is linear})$$

$$\Rightarrow f(x) + f(y) = f_1(x+y) + i f_2(x+y)$$

$$[f_1(x) + i f_2(x)] + [f_1(y) + i f_2(y)] = f_1(x+y) + i f_2(x+y)$$

$$\Rightarrow [f_1(x) + f_1(y)] + i[f_2(x) + f_2(y)] = f_1(x+y) + i f_2(x+y)$$

$\Rightarrow$ Equating real & imaginary parts, we get

$$f_1(x+y) = f_1(x) + f_1(y)$$

$$\& \quad f_2(x+y) = f_2(x) + f_2(y)$$

First condition of linearity is satisfied.

Again, let $\alpha \in C$, $x \in M$. Then

$$f(\alpha x) = f_1(\alpha x) + i f_2(\alpha x)$$

$$\Rightarrow \alpha f(x) = f_1(\alpha x) + i f_2(\alpha x)$$

$$\Rightarrow \alpha[f_1(x) + i f_2(x)] = f_1(\alpha x) + i f_2(\alpha x)$$

Equating real & imaginary parts, we get

$$f_1(\alpha x) = \alpha f_1(x)$$

$$\& \quad f_2(\alpha x) = \alpha f_2(x)$$

Hence f_1 & f_2 are real valued linear functional on M

Also f_1 is bounded as

$$\left.\begin{array}{l} f_1(x) \le \|f\|_M \|x\| \\ \& \ -f_1(x) \le \|f\|_M \|x\| \end{array}\right\} \Rightarrow |f_1(x)| \le \|f\|_M \|x\|$$

$\Rightarrow$ f_1 is bounded

Remark :-

" A complex normed space X can be thought of as a real normed space X_R as such we denoted by X_R^* the dual of X^*, then it is easy to check the function $r : X^* \to X_R^*$ such that
$r(f) = Re(f)$ then r is one to one normed preserving mapping from X^* onto X_R^*.

~~Step 2~~ :- So we have

$$f_1(x) = Re(f(x)) \le |f(x)| \le \|f\|_M \|x\| = p(x). \quad \forall x \in M_R$$

Similarly, we have

$$-f_1(x) \le \|f\|_M \|x\|$$
$$|f_1(x)| \le \|f\|_M \|x\|$$

$\Rightarrow f_1$ is bounded real valued linear functional on M_R and hence there exists a real bounded linear functional g_1 on X_R such that

$$g_1|M_R = f_1 \quad \text{and} \quad \|g_1\|_{X_R} = \|f_1\|_{M_R}$$

and also $\quad g_1(x) \le p(x)$

~~Step 3~~ :- Now, let us define a function such that

Define $g : X \to \mathbb{C}$

$$g(x) = g_1(x) - i\, g_1(ix)$$

we claim that g is linear. Then

Let $x, x_1, x_2 \in X$ and $\lambda \in \mathbb{R}$.

$$g(x_1 + x_2) = g(x_1) + g(x_2)$$

and $\quad g(\alpha x) = \alpha\, g(x)$

and hence g is linear.

Step 2

~~Step 4~~ :- We show that g is a complex linear functional on complex normed space X.

we have $\quad g(x) = g_1(x) - i\, g_1(ix)$. Then

$$g(ix) = g_1(ix) + i\, g_1(x)$$
$$= -i^2 g_1(ix) + i\, g_1(x) \qquad (\because i^2 = -1)$$
$$= i\,[\, g_1(x) - i\, g_1(ix)\,]$$
$$= i\, g(x) \quad \forall x \in X$$

therefore $|g(x)| = \cancel{R}\, e^{-i\theta} g(x)$

$$= Re\left[e^{-i\theta} g(x)\right]$$
$$= Re\left[g(e^{-i\theta}x)\right]$$
$$= g_1(e^{-i\theta}x) \le p(e^{-i\theta}x) \quad (\text{by step } 2)$$
$$= \|f\|_M \, \|e^{-i\theta}x\|$$
$$= \|f\|_M \, \|x\|$$

~~therefore g is bounded and~~

We know that

$$\frac{|g(x)|}{\|x\|} \le \|f\|_M$$

$$\Rightarrow \sup\left\{ \frac{|g(x)|}{\|x\|} : x \in X \right\} \le \|f\|_M$$

$$\Rightarrow \|g\|_X \le \|f\|_M \quad \text{———} \quad (2)$$

Hence from (1) & (2) we have

$$\|g\|_X = \|f\|_M$$

Hence the theorem.

Important Theorems based on Hahn Banach