Dr. S.K. Rana

Deptt. of Mathematics

MSc II Sem

(H-2050)

(Measure and Integration)

REDMI NOTE 8 PRO
AI QUAD CAMERA

Lebesgue Measure and Exterior Measure

Cardinal Number

$A \sim B$

(Equivalence Relation)

$\downarrow$

decomposes equivalence classes

$\downarrow$

Every equivalence class defines a cardinal numbers

Cardinal number = power = potency.

The cardinal number of ϕ is defined as 0.

Cardinal numbers of $P, Q, R, S \ldots$ are denoted by $p, q, r, s \ldots$ respectively.

Cardinal number of A is denoted by $|A|$ or Card A.

So $|P| =$ Card $P = p$,

$|Q| =$ Card $Q = q$.

(Lebesgue Measure of a Set)

Measure of an open set and a closed set :-

Consider the interval (a, b)

Let G be an open set and contained in (a, b)

then the measure of G is denoted by $m(G)$

and has the property

$$m(G) \leq b-a$$

Let F be a closed set and contained in (a, b)

Then the measure of F is defined as

$$m(F) = b-a-m(F') \quad \text{where } F' = (a,b)-F$$

Measure of an open interval —

$$m[(a,b)] = b-a$$

For exp. $m[(5,7)] = 7-5 = 2$. etc.

Oboviously $m(G) > 0$

Measure of rectangle :-

$$R\ (a<x<b;\ c<y<d)$$

The area of a closed rectangle $R\ (a \leq x \leq b,\ c \leq y \leq d)$ is the measure of R defined as

$$m(R) = (b-a)(d-c)$$

Measure of parallelopiped —

$$V\ (a<x<b,\ c<y<d,\ l<z<m)$$

The volume of a closed parallelopiped $V\ (a \leq x \leq b,\ c \leq y \leq d,\ l \leq z \leq m)$ is the measure of R defined as

$$m(V) = (b-a)(d-c)(m-l)$$

REDMI NOTE 8 PRO

Exterior Measure / 67
or (Lebesgue exterior measure)

The exterior measure of any set A, denoted by $m_e(A)$, is defined as:

$$m_e(A) = \inf.\{m(G) : G \text{ is open}, G \supset A\}$$

obeviously $m(G) > m_e(A)$ and for $\epsilon > 0$ $\exists$ an open set G such that $m(G) < m_e(A) + \epsilon$.

Also $m_e(\phi) = 0$.

Interior Measure / 67
or (Lebesgue interior measure)

The interior measure of any set A, denoted by $m_i(A)$, is defined as:

$$m_i(A) = \sup\{m(F) : F \text{ is closed}, F \subset A\}$$

Also remember that

$$m_i(A) = b - a - m_e(A')$$

Also $m_i(A) \geq 0$ and $m_i(\phi) = 0$.

Measurable Set (or Lebesgue Measurable) —

A set A is said to be measurable (or Lebesgue measurable) if

$$m_i(A) = m_e(A)$$

The common value of $m_e(A)$ & $m_i(A)$ is called its measure and is denoted by $m(A)$. Thus, $m_e(A) = m_i(A) = m(A)$ if A is measurable.

Remark: g.l.b. = inf., l.u.b. = Sup.

Thm 6 / 72

If E_1 and E_2 are subsets of $[a,b]$ then

$$m_e(E_1) + m_e(E_2) \geq m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2)$$

and $$m_i(E_1) + m_i(E_2) \leq m_i(E_1 \cup E_2) + m_i(E_1 \cap E_2)$$

Proof :- Let E_1 & E_2 be subsets of $[a,b]$.

Let $\epsilon > 0$ $\exists$ open sets G_1 & G_2 such that

$E_1 \subset G_1$ & $E_2 \subset G_2$ then

$$m(G_1) < m_e(E_1) + \frac{\epsilon}{2}$$

& $$m(G_2) < m_e(E_2) + \frac{\epsilon}{2}$$

$$\Rightarrow m(G_1) + m(G_2) < m_e(E_1) + m_e(E_2) + \epsilon \quad \text{--- (1)}$$

But we know that

$$m(G_1) + m(G_2) = m(G_1 \cup G_2) + m(G_1 \cap G_2) \quad \text{--- (2)}$$

From (1) & (2)

$$m(G_1 \cup G_2) + m(G_1 \cap G_2) < m_e(E_1) + m_e(E_2) + \epsilon$$

Since $G_1 \cup G_2 \subset E_1 \cup E_2$

& $G_1 \cap G_2 \subset E_1 \cap E_2$

then

$$m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2) < m_e(E_1) + m_e(E_2) + \epsilon$$

Since ϵ is arbitrary, so $\epsilon \to 0$.

$$\Rightarrow m_e(E_1 \cup E_2) + m_e(E_1 \cap E_2) < m_e(E_1) + m_e(E_2) \quad \text{Proved}$$

Now Replace $E_1 = E_1'$ & $E_2 = E_2'$

$$\Rightarrow m_e(E_1' \cup E_2') + m_e(E_1' \cap E_2') < m_e(E_1') + m_e(E_2')$$

$$\Rightarrow m_e[(E_1 \cap E_2)'] + m_e[(E_1 \cup E_2)'] < m_e(E_1') + m_e(E_2')$$

where $E_1' = [a,b] - E_1$ & $E_2' = [a,b] - E_2$

by defn of Interior measure,

$$\Rightarrow [b-a-m_i(E_1 \cap E_2)] + [b-a-m_i(E_1 \cup E_2)] \leq [b-a-m_i(E_1)] + [b-a-m_i(E_2)]$$

$$\Rightarrow m_i(E_1 \cap E_2) + m_i(E_1 \cup E_2) \geq m_i(E_1) + m_i(E_2)$$

Proved