Fluid Dynamics - II - 4050 (Maths)

Kelvin's Circulation theorem: It states that the circulation round any closed material curve is invariant in an inviscid fluid provided that the body forces are conservative and the pressure is single valued function of density only.

For a closed material curve C, the circulation Γ is defined as

$$\Gamma = \int_C q \cdot dr$$

or
$$\frac{d\Gamma}{dt} = \frac{d}{dt}\int_C q \cdot dr = \int_C \left[\frac{dq}{dt} dr + q \cdot d\left(\frac{dr}{dt}\right)\right] \quad \text{—— (1)}$$

The Euler's equation of motion is

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p = -\nabla\Omega - \frac{1}{\rho}\nabla p \quad \text{—— (2)}$$

where Ω is the single-valued potential of the conservative body forces.

From (1) and (2), we have

$$\frac{d\Gamma}{dt} = \int_C \left[\left(-\nabla\Omega - \frac{1}{\rho}\nabla p\right)\cdot dr + q \cdot dq\right]$$

or,
$$\frac{d\Gamma}{dt} = \int_C \left[-d\Omega - \frac{dp}{\rho} + q\cdot dq\right], \text{ as } d\left(\frac{dr}{dt}\right) = dq$$

or,
$$\frac{d\Gamma}{dt} = -\int_C d\left[\Omega + \int \frac{dp}{\rho} - \frac{1}{2}q^2\right] \quad \text{—— (3)}$$

This gives the rate of change of flow along any line moving with the fluid. Since the pressure p is a single-valued function of density ρ, the integral in (3) vanishes. Thus, we have

$$\frac{d\Gamma}{dt} = 0 \Rightarrow \Gamma = \text{Constant}.$$

Therefore the circulation Γ is independent of time.

Cor: If any portion of the moving fluid once irrotational under the above conditions, then it remains so for all subsequent times.

Consider the motion is irrotational at

any instant of time t then for every closed curve circulation is zero i.e

$$\Gamma = \int_C q \cdot dr = \iint_S (\nabla \times q) \cdot ds$$

Since the motion is irrotational at all points of S.
Therefore

$$\nabla \times q = 0 \Rightarrow \iint_S (\nabla \times q) \cdot ds = 0$$

$$\Rightarrow \Gamma = 0$$

It follows that the irrotational motion is permanent.

Reference: Fluid Dynamics by Krishna's Educational Publishers, Meerut.