Fluid Dynamics - M-4450 (Maths)

Kelvin's minimum energy theorem:

The irrotational motion of a liquid occupying a singly connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

Since the actual motion is set up by giving impulses on the boundaries, the motion must be irrotational and acyclic, so that a single valued velocity potential ϕ exists such that

$$q = -\nabla\phi \quad \text{—— (1)}$$

Let q_0 be the fluid velocity of any other motion consistent with the same normal velocity of the boundary whose kinetic energy be T_0. Then

$$n.q_0 = n.q \Rightarrow n.(q_0 - q) = 0 \Rightarrow n.u = 0,\ u = q_0 - q \quad \text{—— (2)}$$

Also from the equation of continuity, we have

$$\nabla.q_0 = 0 = \nabla.q \Rightarrow \nabla.(q_0 - q) = 0 \Rightarrow \nabla.u = 0 \quad \text{—— (3)}$$

Here
$$T_0 = \frac{1}{2}\int_V \rho q_0^2\, dv = \frac{1}{2}\int_V \rho (q+u)^2 . dv$$

or,
$$T_0 = \frac{1}{2}\int_V \rho (q^2 + u^2 + 2q.u)\, dv$$

or,
$$T_0 = \frac{1}{2}\int_V \rho q^2\, dv + \frac{1}{2}\int_V \rho u^2 dv + \rho\int_V q.u\, dv$$

or,
$$T_0 = T + \frac{1}{2}\int_V \rho u^2 dv - \rho\int_V (\nabla\phi . u)\, dv$$

where T is the kinetic energy of the irrotational motion of which ϕ is the velocity potential and q is the fluid velocity. The third integral is zero by Gauss theorem.

Thus,
$$T_0 - T = \frac{1}{2}\int_V \rho u^2 dv.$$

But T_0, T are essentially positive quantities so that

$$T_0 - T > 0 \Rightarrow T < T_0 \quad \text{—— (5)}$$

Hence the theorem follows which is due to Lord Kelvin and was subsequently generalised by him so as to apply to all dynamical systems started impulsively from rest.

Reference:- Fluid Dynamics, Krishna Publication, Meerut.