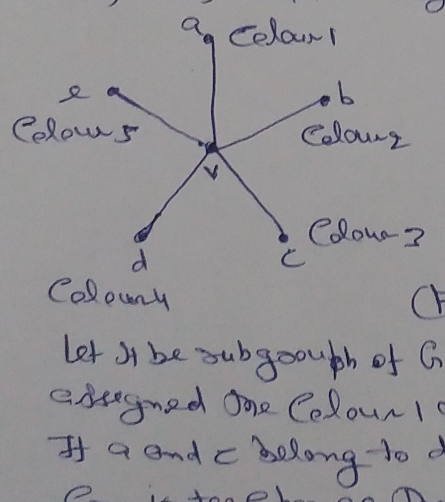
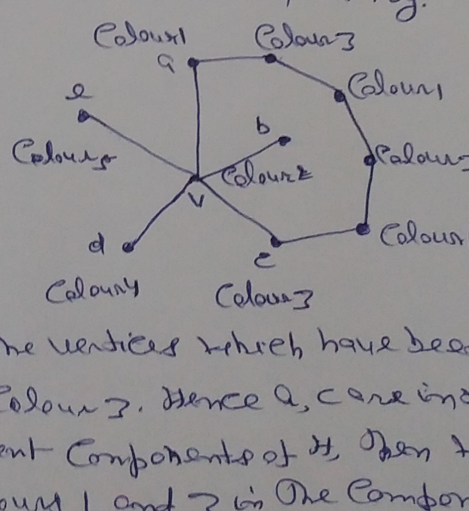
Discrete Mathematics – M-2051 (Maths)

Five Colour Theorem: The vertices of a planar graph can be properly coloured with five colours.

We shall prove the result by induction on the number of vertices of graph G. If the number of vertices n in a graph is less than or equal to 5 then we can properly colour these vertices with five colours. Hence the result holds. We assume that the result holds for all planar graphs having less than n vertices. Now consider a planar graph with n vertices. Then it must have a vertex v with degree five or less. Let $G' = G - v$, the graph obtained from G by deleting vertex v. Then by induction hypothesis, G' (of $n-1$ vertices) requires no more than five colours. Paint the vertices of G' using five colours and now add to it the vertex v along with all edges incident on v. If the degree of v is less than 4, we can assign a proper colour to v and obtain a proper colouring of G.

This leaves only a case when $d(v) = 5$ and all the five colours have been used in colouring the vertices a, b, c, d and e adjacent to v as shown in Fig.





(Fig.)

Let H be subgraph of G on the vertices which have been assigned the Colour 1 and Colour 3. Hence a, c are in H. If a and c belong to different components of H, then we can interchange the colours 1 and 3 in the component which contains vertex a without destroying the proper colouring of $G-v$. This interchange will paint vertices a and c with colour 3. Now colour 1 can be assigned to vertex v and thus G is 5-chromatic. On the other hand, if a and c are in the same component of H, then there is a path P from a to c whose vertices are painted alternately with colours 1 and 3 as shown in Fig. The path P together with the edges (a,v) and (c,v) form a circuit C which encloses b as shown in Fig. Consider now the subgraph K of G on the vertices painted with colours 2 or 4. Since C encloses v_2 or v_4 but not both, vertices b and d belong to different components of K. Therefore we can interchange the colours 2 and 4 in the component containing b without destroying the proper colouring of $G-v$. This interchange will paint vertices b and d with colour 4. Now colour 2 can be assigned to vertex v. Hence the theorem.

Ref: Discrete Mathematics by Krishna's Educational Publishers. Meerut.